Local Time and the Unification of Physics
Part II. Local System

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Abstract: As a continuation of Part I [8], a more precise formulation of local time and local system is given. The observation process is reflected in order to give a relation between the classical physics for centers of mass of local systems and the quantum mechanics inside a local system. The relation will give a unification of quantum mechanics and general relativity in some cases. The existence of local time and local motion is proved so that the stationary nature of the universe is shown to be consistent with the local motion.

V. Definition of Local System

As announced in Part I [8], we begin in this part II with stating a precise definition of local clock and local time, where as we have discussed in Part I, local clock is the local system itself in that every existence is clocking. (See section IV [8], especially recall the statement: In this sense, "clocking" is the natural activity of any local system.) Thus the purpose of this section is the definition of local system.

To do so, we begin with introducing a stationary universe ϕ. By nature what is called the universe must be a closed universe, within which is all. We characterize it by a certain quantum-mechanical condition.

Let $\mathcal{H}$ be a separable Hilbert space, and set

$$\mathcal{U} = \{\phi\} = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{\ell=0}^{\infty} \mathcal{H}^n \right) \quad (\mathcal{H}^n = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ factors}}).$$

$\mathcal{U}$ is called a Hilbert space of possible universes. An element ϕ of $\mathcal{U}$ is called a universe and is of the form of an infinite matrix $(\phi_{n\ell})$ with components $\phi_{n\ell} \in \mathcal{H}^n$. $\phi = 0$ means $\phi_{n\ell} = 0$ for all n, ℓ.

Let $\mathcal{O} = \{S\}$ be the totality of the selfadjoint operators S in $\mathcal{U}$ of the form $S\phi = (S_{n\ell}\phi_{n\ell})$ for $\phi = (\phi_{n\ell}) \in \mathcal{D}(S) \subset \mathcal{U}$, where each component $S_{n\ell}$ is a selfadjoint operator in $\mathcal{H}^n$. Our characterization of the universe ϕ is the following condition.

Axiom 1. There is a selfadjoint operator $H \in \mathcal{O}$ in $\mathcal{U}$ such that for some $\phi \in \mathcal{U} - \{0\}$ and $\lambda \in \mathbf{R}$

$$H\phi \approx \lambda\phi \tag{1}$$

in the following sense: Let F_n be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_n)(=$ the number of elements in $F_n) = n$ and let $\{F_n^\ell\}_{\ell=0}^\infty$ be the totality of such F_n (note: the set $\{F_n^\ell\}_{\ell=0}^\infty$ is countable). Then the formula (1) in the above means that there are integral sequences $\{n_k\}_{k=1}^\infty$ and $\{\ell_k\}_{k=1}^\infty$ and a real sequence $\{\lambda_{n_k\ell_k}\}_{k=1}^\infty$ such that $F_{n_k}^{\ell_k} \subset F_{n_{k+1}}^{\ell_{k+1}}$; $\bigcup_{k=1}^\infty F_{n_k}^{\ell_k} = \mathbf{N}$;

$$H_{n_k\ell_k}\phi_{n_k\ell_k} = \lambda_{n_k\ell_k}\phi_{n_k\ell_k}, \quad \phi_{n_k\ell_k} \neq 0, \quad k = 1, 2, 3, \cdots; \tag{2}$$

and

$$\lambda_{n_k\ell_k} \to \lambda \quad \text{as} \quad k \to \infty.$$

H is an infinite matrix $(H_{n\ell})$ of selfadjoint operators $H_{n\ell}$ in $\mathcal{H}^n$. Axiom 1 asserts that this matrix converges in the sense of (1) on our universe ϕ. We remark that our universe ϕ is not determined uniquely by this condition.

The universe as a state ϕ is a whole, within which is all. As such a whole, the state ϕ can follow the two ways: The one is that ϕ develops along a global time T in the grand universe $\mathcal{U}$ under a propagation $\exp(-iTH)$, and another is that ϕ is a bound state of H. If there were such a global time T as in the first case, all phenomena had to develop along that global time T, and the locality of time would be lost. We could then *not* construct a notion of local times compatible with general theory of relativity. The only one possibility is therefore to adopt the stationary universe ϕ of Axiom 1.

The following axiom asserts the existence of configuration and momentum operators and that the canonical commutation relation between them holds. This is a basis of our definition of time, where configuration and momentum are given first, and then local times are defined in each local system of finite number of quantum-mechanical particles.

Axiom 2. Let $n \geq 1$ and F_{n+1} be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_{n+1}) = n + 1$. Then for any $j \in F_{n+1}$, there are selfadjoint operators $X_j = (X_{j1}, X_{j2}, X_{j3})$ and $P_j = (P_{j1}, P_{j2}, P_{j3})$ in $\mathcal{H}^n$, and constants $m_j > 0$ such that

$$[X_{j\ell}, X_{km}] = 0, \quad [P_{j\ell}, P_{km}] = 0, \quad [X_{j\ell}, P_{km}] = i\delta_{jk}\delta_{\ell m},$$

$$\sum_{j \in F_{n+1}} m_j X_j = 0, \quad \sum_{j \in F_{n+1}} P_j = 0.$$

The Stone-von Neumann theorem and Axiom 2 specify the space dimension (see [1], p.452) as 3 dimension. We identify $\mathcal{H}^n$ with $L^2(R^{3n})$ in the following.

What we intend to mean by the (n, ℓ)-th component $H_{n\ell}$ $(n, \ell \geq 0)$ of $H = (H_{n\ell})$ in Axiom 1 is the usual $N = n+1$ body Hamiltonian with center of mass removed to accord to the requirement $\sum_{j \in F_{n+1}} m_j X_j = 0$ in Axiom 2. For the local Hamiltonian $H_{n\ell}$ we thus make the following postulate.

Axiom 3. Let $n \geq 0$ and F_N $(N = n + 1)$ be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_N) = N$. Let $\{F_N^\ell\}_{\ell=0}^\infty$ be the countable totality of such F_N. Then the component Hamiltonian $H_{n\ell}$ $(\ell \geq 0)$ of H in Axiom 1 is of the form

$$H_{n\ell} = H_{n\ell 0} + V_{n\ell}, \quad V_{n\ell} = \sum_{\substack{\alpha = (i,j) \\ 1 \leq i < j < \infty, \ i,j \in F_N^\ell}} V_\alpha(x_\alpha)$$

on $C_0^\infty(\mathbf{R}^{3n})$, where $x_\alpha = x_i - x_j$ $(\alpha = (i,j))$ with x_i being the position vector of the i-th particle, and $V_\alpha(x_\alpha)$ is a real-valued measurable function of $x_\alpha \in \mathbf{R}^3$ which is $H_{n\ell 0}$-bounded with $H_{n\ell 0}$-bound of $V_{n\ell}$ less than 1. $H_{n\ell 0} = H_{(N-1)\ell 0}$ is the free Hamiltonian of the N-particle system, which has the form like

$$\sum_{\ell=1}^n \sum_{k=1}^3 \frac{1}{2\mu_\ell} \frac{\partial^2}{\partial x_{\ell k}^2} \quad \text{with } \mu_\ell > 0 \text{ being reduced mass.}$$

This axiom implies that $H_{n\ell} = H_{(N-1)\ell}$ is uniquely extended to a selfadjoint operator bounded from below in $\mathcal{H}^n = \mathcal{H}^{N-1} = L^2(\mathbf{R}^{3(N-1)})$ by the Kato-Rellich theorem.

We do not include vector potentials in the Hamiltonian $H_{n\ell}$ of Axiom 3, for we take the position that what is elementary is the electronic charge, and the magnetic forces are the consequence of the motions of charges.

Let P_H denote the orthogonal projection onto the space of bound states for a selfadjoint operator H. We call the set of all states orthogonal to the space of bound states a scattering space, and its element as a scattering state. Let $\phi = (\phi_{n\ell})$ with $\phi_{n\ell} = \phi_{n\ell}(x_1, \cdots, x_n) \in L^2(R^{3n})$ be the universe in Axiom 1, and let $\{n_k\}$ and $\{\ell_k\}$ be the sequences specified there. Let $x^{(n,\ell)}$ denote the relative coordinates of $n + 1$ particles in F_{n+1}^ℓ.

Definition 1.

(1) We define $\mathcal{H}_{n\ell}$ as the sub-Hilbert space of $\mathcal{H}^n$ generated by the functions $\phi_{n_k \ell_k}(x^{(n,\ell)}, y)$ of $x^{(n,\ell)} \in \mathbf{R}^{3n}$ with regarding $y \in \mathbf{R}^{3(n_k - n)}$ as a parameter, where k moves over a set $\{k \mid n_k \geq n, F_{n+1}^\ell \subset F_{n_k+1}^{\ell_k}, k \in \mathbf{N}\}$.

(2) $\mathcal{H}_{n\ell}$ is called a *local universe* of ϕ.

(3) $\mathcal{H}_{n\ell}$ is said to be non-trivial if $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$.

The total universe ϕ is a single element in $\mathcal{U}$. The local universe $\mathcal{H}_{n\ell}$ can be richer and may have elements more than one. This is because we consider the subsystems of the universe consisting of a finite number of particles. These subsystems receive the

influence from the other particles of infinite number outside the subsystems, and may vary to constitute a non-trivial subspace $\mathcal{H}_{n\ell}$. We will return to this point in section VII.

We can now define local system.

Definition 2.

(1) The restriction of H to $\mathcal{H}_{n\ell}$ is also denoted by the same notation $H_{n\ell}$ as the (n, ℓ)-th component of H.

(2) We call the pair $(H_{n\ell}, \mathcal{H}_{n\ell})$ a local system.

(3) The unitary group $e^{-itH_{n\ell}}$ $(t \in \mathbf{R}^1)$ on $\mathcal{H}_{n\ell}$ is called the *proper clock* of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $\mathcal{H}_{n\ell}$ is non-trivial: $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$. (Note that the clock is defined only for $N = n + 1 \geq 2$, since $H_{0\ell} = 0$ and $P_{H_{0\ell}} = I$.)

(4) The universe ϕ is called *rich* if $\mathcal{H}_{n\ell}$ equals $\mathcal{H}^n = L^2(\mathbf{R}^{3n})$ for all $n \geq 1$, $\ell \geq 0$. For a rich universe ϕ, $H_{n\ell}$ equals the (n, ℓ)-th component of H.

Definition 3.

(1) The parameter t in the exponent of the proper clock $e^{-itH_{n\ell}} = e^{-itH_{(N-1)\ell}}$ of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is called the (quantum-mechanical) *proper time* or *local time* of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$.

(2) This time t is denoted by $t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ indicating the local system under consideration.

This definition is a one reverse to the usual definition of the motion or dynamics of the N-body quantum systems, where the time t is given *a priori* and then the motion of the particles is defined by $e^{-itH_{(N-1)\ell}}f$ for a given initial state f of the system.

Time is thus defined only for local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and is determined by the associated proper clock $e^{-itH_{n\ell}}$. Therefore there are infinitely many number of times $t = t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ each of which is proper to the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$. In this sense time is a local notion. There is no time for the total universe ϕ in Axiom 1, which is a bound state of the total Hamiltonian H in the sense of the condition (1) of Axiom 1.

To see the meaning of our definition of time, we quote a theorem from [3]. To state the theorem we introduce some notations concerning the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, assuming that the universe ϕ is rich: Let $b = (C_1, \cdots, C_{\sharp(b)})$ be a decomposition of the set $\{1, 2, \cdots, N\}$ $(N = n+1)$ into $\sharp(b)$ number of disjoint subsets $C_1, \cdots, C_{\sharp(b)}$ of $\{1, 2, \cdots, N\}$. b is called a cluster decomposition. $H_b = H_{n\ell,b} = H_{n\ell} - I_b = H_{n\ell}^b + T_{n\ell,b} = H^b + T_b$ is the truncated Hamiltonian for the cluster decomposition b with $1 \leq \sharp(b) \leq N$, where I_b is the sum of the intercluster interactions between various two different clusters in b, and T_b is the sum of the intercluster free energies among various clusters in b. $x_b \in \mathbf{R}^{3(\sharp(b)-1)}$ is the intercluster coordinates among the centers of mass of the clusters in b, while $x^b \in \mathbf{R}^{3(N-\sharp(b))}$ denotes the intracluster coordinates inside the clusters of b so that $x \in \mathbf{R}^{3n} = \mathbf{R}^{3(N-1)}$ is expressed as $x = (x_b, x^b)$. Note that x^b is decomposed as $x^b = (x_1^b, \cdots, x_{\sharp(b)}^b)$, where

4

each $x_j^b \in \mathbf{R}^{3(\sharp(C_j)-1)}$ is the internal coordinate of the cluster C_j, describing the configuration of the particles inside C_j. The operator H^b is accordingly decomposed as $H^b = H_1 + \cdots + H_{\sharp(b)}$, and each component H_j is defined in the space $\mathcal{H}_j^b = L^2(\mathbf{R}_{x_j^b}^{3(\sharp(C_j)-1)})$, whose tensor product $\mathcal{H}_1^b \otimes \cdots \otimes \mathcal{H}_{\sharp(b)}^b$ is the internal state space $\mathcal{H}^b = L^2(\mathbf{R}_{x^b}^{3(N-\sharp(b))})$. The free energy T_b is defined in the external space $\mathcal{H}_b = L^2(\mathbf{R}_{x_b}^{3(\sharp(b)-1)})$, and the truncated Hamiltonian $H_b = H^b + T_b = I \otimes H^b + T_b \otimes I$ is defined in the total space $\mathcal{H}_{n\ell} = \mathcal{H}_b \otimes \mathcal{H}^b = L^2(\mathbf{R}_x^{3(N-1)})$. v_b is the velocity operator conjugate to the intercluster coordinates x_b. $P_b = P_{H^b}$ is the eigenprojection associated with the subsystem H^b of H, i.e. the orthogonal projection onto the eigenspace of H^b, which is defined in $\mathcal{H}^b$ and extended as $I \otimes P_b$ to the total space $\mathcal{H}_{n\ell}$. P_b^M is the M-dimensional partial projection of this eigenprojection P_b. We define for a k-dimensional multi-index $M = (M_1, \cdots, M_k)$, $M_j \geq 1$ and $k = 1, \cdots, N-1$,

$$\widehat{P}_k^M = \left(I - \sum_{\sharp(b)=k} P_b^{M_k}\right) \cdots \left(I - \sum_{\sharp(d)=2} P_d^{M_2}\right) (I - P^{M_1}),$$

where note that $P^{M_1} = P_a^{M_1} = P_H^{M_1}$ for $\sharp(a) = 1$ is uniquely determined. We also define for a $\sharp(b)$-dimensional multi-index $M_b = (M_1, \cdots, M_{\sharp(b)-1}, M_{\sharp(b)}) = (\widehat{M}_b, M_{\sharp(b)})$

$$\widetilde{P}_b^{M_b} = P_b^{M_{\sharp(b)}} \widehat{P}_{\sharp(b)-1}^{\widehat{M}_b}, \quad 2 \leq \sharp(b) \leq N.$$

It is clear that

$$\sum_{2 \leq \sharp(b) \leq N} \widetilde{P}_b^{M_b} = I - P^{M_1},$$

provided that the component M_k of M_b depends only on the number k but not on b. In the following we use such M_b's only. Under these circumstances, the following is known to hold.

Theorem 1 ([3]) *Let $N = n + 1 \geq 2$ and let $H_{N-1} = H_{n\ell}$ be the Hamiltonian for a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$. Let suitable conditions on the decay rate for the pair potentials $V_{ij}(x_{ij})$ be satisfied (see, e.g., Assumption 1 in [7]). Let $\||x^a|^2 P_a^M\| < \infty$ be satisfied for any integer $M \geq 1$ and cluster decomposition a with $2 \leq \sharp(a) \leq N-1$. Let $f \in \mathcal{H}^{N-1}$. Then there is a sequence $t_m \to \pm\infty$ (as $m \to \pm\infty$) and a sequence M_b^m of multi-indices whose components all tend to ∞ as $m \to \pm\infty$ such that for all cluster decompositions b, $2 \leq \sharp(b) \leq N$, and $\varphi \in C_0^\infty(\mathbf{R}_{x_b}^{3(\sharp(b)-1)})$*

$$\|\{\varphi(x_b/t_m) - \varphi(v_b)\}\widetilde{P}_b^{M_b^m} e^{-it_m H_{N-1}} f\| \to 0 \tag{3}$$

as $m \to \pm\infty$.

The asymptotic relation (3) roughly means that, if we restrict our attention to the part $\widetilde{P}_b^{M_b^m}$ of the evolution $e^{-it H_{N-1}} f$, in which the particles inside any cluster of b are bounded

while any two different clusters of b are scattered, then the magnitude of quantum-mechanical velocity $v_b = m_b^{-1} p_b$, where m_b is some diagonal mass matrix, is approximated by the square root of a classical value

$$|v_b^{(c)}|^2 = \lim_{m \to \pm \infty} (|v_b|^2 \widetilde{P}_b^{M_b^m} e^{-it_m H_{N-1}} f, \widetilde{P}_b^{M_b^m} e^{-it_m H_{N-1}} f)$$

asymptotically as $m \to \pm \infty$ and the local time t of the N body system $H_{N-1} = H_{n\ell}$ is asymptotically equal to the quotient of the configuration by the velocity of the scattered particles (or clusters, exactly speaking):

$$\frac{|x_b|}{\left|v_b^{(c)}\right|} \tag{4}$$

on the evolving state $\widetilde{P}_b^{M_b^m} e^{-it_m H_{N-1}} f$. This means by $v_b = m_b^{-1} p_b$ that the local time t is asymptotically and approximately measured if the values of the configurations and momenta for the scattered particles of the local system $(H_{N-1}, \mathcal{H}_{N-1}) = (H_{n\ell}, \mathcal{H}_{n\ell})$ are given.

We note that the time measured by (4) is independent of the choice of cluster decomposition b according to Theorem 1. This means that t can be taken as a common parameter of motion inside the local system, and can be called *time* of the local system in accordance with the notion of 'common time' in Newton's sense: "relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, $\cdots$" ([12], p.6). Once we take t as our notion of time for the system $(H_{n\ell}, \mathcal{H}_{n\ell})$, t recovers the usual meaning of time, by the identity for $e^{-itH_{n\ell}} f$ known as the Schrödinger equation:

$$\left(\frac{1}{i} \frac{d}{dt} + H_{n\ell} \right) e^{-itH_{n\ell}} f = 0.$$

Time $t = t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ is a notion defined only in relation with the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$. To other local system $(H_{mk}, \mathcal{H}_{mk})$, there is associated other local time $t_{(H_{mk}, \mathcal{H}_{mk})}$, and between $t = t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ and $t_{(H_{mk}, \mathcal{H}_{mk})}$, there is no relation, and they are completely independent notions. In other words, $\mathcal{H}_{n\ell}$ and $\mathcal{H}_{mk}$ are different spaces unless $n = m$ and $\ell = k$. And even when the two local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$ have a non-vanishing common part: $F_{n+1}^\ell \cap F_{m+1}^k \neq \emptyset$, the common part constitutes its own local system $(H_{pj}, \mathcal{H}_{pj})$, and its local time cannot be compared with those of the two bigger systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$, because these three systems have different base spaces, Hamiltonians, and clocks. More concretely speaking, the times are measured through the quotients (4) for each system. But the L^2-representations of the base Hilbert spaces $\mathcal{H}_{n\ell}, \mathcal{H}_{mk}, \mathcal{H}_{pj}$ for those systems are different unless they are identical with each other, and the quotient (4) has incommensurable meaning among these representations.

In this sense, local systems are independent mutually. Also they cannot be decomposed into pieces in the sense that the decomposed pieces constitute different local systems.

VI. Relativity as Observation

We now see how we can combine relativity and quantum mechanics in our formulation.

VI.1. Relativity

We note that the center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is always at the origin of the space coordinate system $x_{(H_{n\ell}, \mathcal{H}_{n\ell})} \in \mathbf{R}^3$ for the local system by the requirement: $\sum_{j \in F_{n+1}} m_j X_j = 0$ in Axiom 2, and that the space coordinate system describes just the relative motions inside a local system by our formulation. The center of mass of a local system, therefore, cannot be identified from the local system itself, except the fact that it is at the origin of the coordinates.

Moreover, just as we have seen in the previous section, we know that, not only the time coordinates $t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ and $t_{(H_{mk}, \mathcal{H}_{mk})}$, but also the space coordinates $x_{(H_{n\ell}, \mathcal{H}_{n\ell})} \in \mathbf{R}^3$ and $x_{(H_{mk}, \mathcal{H}_{mk})} \in \mathbf{R}^3$ of these two local systems are independent mutually. Thus the space-time coordinates $(t_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ and $(t_{(H_{mk}, \mathcal{H}_{mk})}, x_{(H_{mk}, \mathcal{H}_{mk})})$ are independent between two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$. In particular, insofar as the systems are considered as quantum-mechanical ones, there is no relation between their centers of mass. In other words, the center of mass of any local system cannot be identified by other local systems quantum-mechanically.

Summing these two considerations, we conclude:

(1) The center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ cannot be identified *quantum-mechanically* by any local system $(H_{mk}, \mathcal{H}_{mk})$ including the case $(H_{mk}, \mathcal{H}_{mk}) = (H_{n\ell}, \mathcal{H}_{n\ell})$.

(2) There is no *quantum-mechanical* relation between any two local coordinates $(t_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ and $(t_{(H_{mk}, \mathcal{H}_{mk})}, x_{(H_{mk}, \mathcal{H}_{mk})})$ of two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$.

Utilizing these properties of the centers of mass and the coordinates of local systems, we may make any postulates concerning

(1) the motions of the *centers of mass* of various local systems,

and

(2) the relation between two local coordinates of any two local systems.

In particular, we may impose *classical* postulates on them as far as the postulates are consistent in themselves.

Thus we assume an arbitrary but fixed transformation:

$$y_2 = f_{21}(y_1) \tag{5}$$

between the coordinate systems $y_j = (y_j^\mu)_{\mu=0}^3 = (y_j^0, y_j^1, y_j^2, y_j^3) = (ct_j, x_j)$ for $j = 1, 2$, where c is the speed of light in vacuum and (t_j, x_j) is the space-time coordinates of

the local system $L_j = (H_{n_j \ell_j}, \mathcal{H}_{n_j \ell_j})$. We regard these coordinates $y_j = (ct_j, x_j)$ as *classical* coordinates, when we consider the motions of centers of mass and the relations of coordinates of various local systems. We can now postulate the general principle of relativity on the physics of the centers of mass:

Axiom 4. The laws of physics which control the relative motions of the centers of mass of local systems are covariant under the change of the coordinates from $(ct_{(H_{mk}, \mathcal{H}_{mk})}, x_{(H_{mk}, \mathcal{H}_{mk})})$ to $(ct_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ of the reference frame local systems for any pair $(H_{mk}, \mathcal{H}_{mk})$ and $(H_{n\ell}, \mathcal{H}_{n\ell})$ of local systems.

We note that this axiom is consistent with the Euclidean metric adopted for the quantum-mechanical coordinates inside a local system, because Axiom 4 is concerned with classical motions of the centers of mass *outside* local systems, and we are dealing here with a different aspect of nature from the quantum-mechanical one *inside* a local system.

Axiom 4 implies the invariance of the distance under the change of coordinates between two local systems. Thus the metric tensor $g_{\mu\nu}(ct, x)$ which appears here satisfies the transformation rule:

$$g^1_{\mu\nu}(y_1) = g^2_{\alpha\beta}(f_{21}(y_1)) \frac{\partial f^\alpha_{21}}{\partial y_1^\mu}(y_1) \frac{\partial f^\beta_{21}}{\partial y_1^\nu}(y_1), \tag{6}$$

where $y_1 = (ct_1, x_1)$; $y_2 = f_{21}(y_1)$ is the transformation (5) in the above from $y_1 = (ct_1, x_1)$ to $y_2 = (ct_2, x_2)$; and $g^j_{\mu\nu}(y_j)$ is the metric tensor expressed in the classical coordinates $y_j = (ct_j, x_j)$ for $j = 1, 2$.

The second postulate is the principle of equivalence, which asserts that the classical coordinate system $(ct_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ is a local Lorentz system of coordinates, insofar as it is concerned with the classical behavior of the center of mass of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$:

Axiom 5. The metric or the gravitational tensor $g_{\mu\nu}$ for the center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ in the coordinates $(ct_{(H_{n\ell}, \mathcal{H}_{n\ell})}, x_{(H_{n\ell}, \mathcal{H}_{n\ell})})$ of itself are equal to $\eta_{\mu\nu}$, where $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, $= 1$ for $\mu = \nu = 1, 2, 3$, and $= -1$ for $\mu = \nu = 0$.

Since, at the center of mass, the classical space coordinates $x = 0$, Axiom 5 together with the transformation rule (6) in the above yields

$$g^1_{\mu\nu}(f^{-1}_{21}(ct_2, 0)) = \eta_{\alpha\beta} \frac{\partial f^\alpha_{21}}{\partial y_1^\mu}(f^{-1}_{21}(ct_2, 0)) \frac{\partial f^\beta_{21}}{\partial y_1^\nu}(f^{-1}_{21}(ct_2, 0)). \tag{7}$$

Also by the same reason, the relativistic proper time $d\tau = \sqrt{-g_{\mu\nu}(ct, 0)dy^\mu dy^\nu}$ $= \sqrt{-\eta_{\mu\nu}dy^\mu dy^\nu}$ at the origin of a local system is equal to c times the quantum-mechanical proper time dt of the system.

By the fact that the classical Axioms 4 and 5 of physics are imposed on the centers of mass which are uncontrollable quantum-mechanically, and on the relation between the coordinates of different, therefore quantum-mechanically non-related local systems, the

consistency of classical relativistic Axioms 4 and 5 with quantum-mechanical Axioms 1–3 is clear:

Theorem 2 *Axioms 1 to 5 are consistent.*

VI.2. Observation

Thus far, we did not mention any about the physics which is actually observed. We have just given two aspects of nature which are mutually independent. We will introduce a procedure which yields what we observe when we see nature. This procedure will not be contradictory with the two aspects of nature which we have discussed, as the procedure is concerned solely with "*how nature looks, at the observer,*" i.e. it is solely concerned with "*at the place of the observer, how nature looks,*" with some abuse of the word "place." The validity of the procedure should be judged merely through the comparison between the observation and the prediction given by our procedure.

We note that we can observe only a finite number of disjoint systems, say $L_1, \cdots, L_k$ with $k \geq 1$ a finite integer. We cannot grasp an infinite number of systems at a time. Further each system L_j must have only a finite number of elements by the same reason. Thus these systems $L_1, \cdots, L_k$ may be identified with local systems in the sense of section V.

Local systems are quantum-mechanical systems, and their coordinates are confined to their insides insofar as we appeal to Axioms 1–3. However we postulated Axioms 4 and 5 on the classical aspects of those coordinates, which make the local coordinates of a local system a classical reference frame for the centers of mass of other local systems. This leaves us the room to define observation as the *classical* observation of the centers of mass of local systems $L_1, \cdots, L_k$. We call this an observation of $L = (L_1, \cdots, L_k)$ inquiring into sub-systems $L_1, \cdots, L_k$, where L is a local system consisting of the particles which belong to one of the local systems $L_1, \cdots, L_k$.

When we observe the sub-local systems $L_1, \cdots, L_k$ of L, we observe the relations or motions among these sub-systems. Internally the local system L behaves following the Hamiltonian H_L associated to the local system L. However the actual observation differs from what the pure quantum-mechanical calculation gives for the system L. For example, when an electron is scattered by a nucleus with relative velocity close to that of light, the observation is different from the pure quantum-mechanical prediction.

The quantum-mechanical process inside the local system L is described by the evolution

$$\exp(-it_L H_L)f,$$

where f is the initial state of the system and t_L is the local time of the system L. The Hamiltonian H_L is decomposed as follows in virtue of the local Hamiltonians $H_1, \cdots, H_k$, which correspond to the sub-local systems $L_1, \cdots, L_k$:

$$H_L = H^b + T + I, \quad H^b = H_1 + \cdots + H_k.$$

Here $b = (C_1, \cdots, C_k)$ is the cluster decomposition corresponding to the decomposition $L = (L_1, \cdots, L_k)$ of L; $H^b = H_1 + \cdots + H_k$ is the sum of the internal energies H_j inside L_j, and is an operator defined in the internal state space $\mathcal{H}^b = \mathcal{H}_1^b \otimes \cdots \otimes \mathcal{H}_k^b$; $T = T_b$ denotes the intercluster free energy among the clusters $C_1, \cdots, C_k$ defined in the external state space $\mathcal{H}_b$; and $I = I_b = I_b(x) = I_b(x_b, x^b)$ is the sum of the intercluster interactions between various two different clusters in the cluster decomposition b (cf. the explanation after Definition 3 in section V).

The main concern in this process would be the case that the clusters $C_1, \cdots, C_k$ form asymptotically bound states as $t_L \to \infty$, since other cases are hard to be observed along the process when as usual the observer's concern is upon the final state of the *bound* sub-systems $L_1, \cdots, L_k$.

The evolution $\exp(-it_L H_L)f$ then behaves asymptotically as $t_L \to \infty$ as follows for some bound states $g_1, \cdots, g_k$ $(g_j \in \mathcal{H}_j^b)$ of local Hamiltonians $H_1, \cdots, H_k$ and for some g_0 belonging to the external state space $\mathcal{H}_b$:

$$\exp(-it_L H_L)f \sim \exp(-it_L h_b)g_0 \otimes \exp(-it_L H_1)g_1 \otimes \cdots \otimes \exp(-it_L H_k)g_k, \quad k \geq 1, \quad (8)$$

where $h_b = T_b + I_b(x_b, 0)$. It is easy to see that $g = g_0 \otimes g_1 \otimes \cdots \otimes g_k$ is given by

$$g = g_0 \otimes g_1 \otimes \cdots \otimes g_k = \Omega_b^{+*}f = P_b \Omega_b^{+*}f,$$

provided that the decomposition of the evolution $\exp(-it_L H_L)f$ is of the simple form as in (8). Here Ω_b^{+*} is the adjoint operator of a canonical wave operator ([2]) corresponding to the cluster decomposition b:

$$\Omega_b^+ = s\text{-}\lim_{t \to \infty} \exp(itH_L) \cdot \exp(-ith_b) \otimes \exp(-itH_1) \otimes \cdots \otimes \exp(-itH_k)P_b,$$

where P_b is the eigenprojection onto the eigenspace of the Hamiltonian $H^b = H_1 + \cdots + H_k$. The process (8) just describes the quantum-mechanical process inside the local system L, and does not specify any meaning related with observation up to the present stage.

To see what we observe at actual observations, let us reflect a process of observation of scattering phenomena. We note that the observation of scattering phenomena is concerned with their initial and final stages by what the scattering itself means. At the final stage of observation of scattering processes, the quantities observed are firstly the points hit by the scattered particles on the screen stood against them. If the circumstances are properly set up, one can further indicate the momentum of the scattered particles at the final stage to the extent that the uncertainty principle allows. Consider, e.g., a scattering process of an electron by a nucleus. Given the magnitude of initial momentum of an electron relative to the nucleus, one can infer the magnitude of momentum of the electron at the final stage as being equal to the initial one by the law of conservation of energy, since the electron and the nucleus are far away at the initial and final stages so that the potential energy between them can be neglected compared to the relative kinetic energy. The direction of momentum at the final stage can also be indicated, up to the error due to the uncertainty principle, by setting a sequence of slits toward the desired direction at each point on the screen so that the observer can detect only the electrons scattered to that direction. The

magnitude of momentum at initial stage can be selected in advance by applying a uniform magnetic field to the electrons, perpendicularly to their momenta, so that they circulate around circles with the radius proportional to the magnitude of momentum, and then by setting a sequence of slits midst the stream of those electrons. The selection of magnitude of initial momentum makes the direction of momentum ambiguous due to the uncertainty principle, since the sequence of slits lets the position of electrons accurate to some extent. To sum up, the sequences of slits at the initial and final stages necessarily require to take into account the uncertainty principle so that some ambiguity remains in the observation.

However, in the actual observation of a *single* particle, we *have to decide* at which point on the screen the particle hits and which momentum the particle has, using the prepared apparatus like the sequence of slits located at each point on the screen. Even if we impose an interval for the observed values, we *have to assume* that the boundaries of the interval are sharply designated. These are the assumption which we always impose on "observations" implicitly. That is to say, we idealize the situation in any observation or in any measurement of a single particle so that the observed values for each particle are sharp for both of the configuration and momentum. In this sense, the values observed actually for each particle must be classical, where the *a priori* indefiniteness and errors associated with any measurement are all included. We have then necessary and sufficient conditions to make predictions about the differential cross section, as we will see in section VI.2.1.

Summarizing, we observe just the classical quantities for each particle at the final stage of all observations. In other words, even if we cannot know the values actually, we have to *presuppose* that the values observed for each particle have sharp values, where all errors associated with measurement are included. We can apply to this fact the remark stated in the third paragraph of this section about the possibility of defining observation as that of the *classical* centers of mass of local systems, and may assume that the actually observed values follow the classical Axioms 4 and 5. Those sharp values actually observed for each particle give, when summed over the large number of particles, the probabilistic nature of physical phenomena, i.e. that of scattering phenomena.

Theoretically, the quantum-mechanical, probabilistic nature of scattering processes is described by differential cross section, defined as the square of the absolute value of the scattering amplitude gotten from scattering operators $S_{bd} = W_b^{+*} W_d^-$, where $W_b^{\pm}$ are usual wave operators. Given the magnitude of the initial momentum of the incoming particle and the scattering angle, the differential cross section gives a prediction about the probability at which point and to which direction on the screen each particle hits on the average. However, as we have remarked, the idealized point on the screen hit by each particle and the scattering angle given as an idealized difference between the directions of the initial and final momenta of each particle have sharp values, and the observation at the final stage is *classical*. We are then required to correct these classical observations by taking into account the classical relativistic effects with those classical quantities, e.g., with the configuration and the momentum of each particle.

VI.2.1. As the first step of the relativistic modification of the scattering process, we consider the scattering amplitude $\mathcal{S}(E, \theta)$, where E denotes the energy level of the scat-

tering process and θ is a parameter describing the direction of the scattered particles. Following our remark made in the previous paragraph, we make the following postulate on the scattering amplitude observed in actual experiment:

Axiom 6.1′. When one observes the final stage of scattering phenomena, the total energy E of the scattering process should be regarded as a classical quantity and is replaced by a relativistic quantity, which obeys the relativistic change of coordinates from the scattering system to the observer's system.

Since it is not known much about $\mathcal{S}(E, \theta)$ in the many body case, we consider an example of the two body case. Consider a scattering phenomenon of an electron by a Coulomb potential Ze^2/r, where Z is a real number, $r = |x|$, and x is the position vector of the electron relative to the scatterer. We assume that the mass of the scatterer is large enough compared to that of the electron and that $|Z|/137 \ll 1$. Then quantum mechanics gives the differential cross section in a Born approximation:

$$\frac{d\sigma}{d\Omega} = |\mathcal{S}(E, \theta)|^2 \approx \frac{Z^2 e^4}{16E^2 \sin^4(\theta/2)},$$

where θ is the scattering angle and E is the total energy of the system of the electron and the scatterer. We assume that the observer is stationary with respect to the center of mass of this system of an electron and the scatterer. Then, since the electron is far away from the scatterer after the scattering and the mass of the scatterer is much larger than that of the electron, we may suppose that the energy E in the formula in the above can be replaced by the *classical* kinetic energy of the electron by Axiom 6.1′. Then, assuming that the speed v of the electron relative to the observer is small compared to the speed c of light in vacuum and denoting the rest mass of the electron by m, we have by Axiom 6.1′ that E is observed to have the following relativistic value:

$$E' = c\sqrt{p^2 + m^2 c^2} - mc^2 = \frac{mc^2}{\sqrt{1 - (v/c)^2}} - mc^2 \approx \frac{mv^2}{2\sqrt{1 - (v/c)^2}},$$

where $p = mv/\sqrt{1 - (v/c)^2}$ is the relativistic momentum of the electron. Thus the differential cross section should be observed approximately equal to

$$\frac{d\sigma}{d\Omega} \approx \frac{Z^2 e^4}{4m^2 v^4 \sin^4(\theta/2)}(1 - (v/c)^2). \tag{9}$$

This coincides with the usual relativistic prediction obtained from the Klein-Gordon equation by a Born approximation. See [6], p.297, for a case which involves the spin of the electron.

Before proceeding to the inclusion of gravity in the general k cluster case, we review this two body case. We note that the two body case corresponds to the case $k = 2$, where L_1 and L_2 consist of single particle, therefore the corresponding Hamiltonians H_1 and H_2 are zero operators on $\mathcal{H}^0 = \mathbf{C} =$ the complex numbers. The scattering

amplitude $\mathcal{S}(E,\theta)$ in this case is an integral kernel of the scattering matrix $\widehat{S} = \mathcal{F}S\mathcal{F}^{-1}$, where $S = W^{+*}W^{-}$ is a scattering operator; $W^{\pm} = s\text{-}\lim_{t\to\pm\infty}\exp(itH_L)\exp(-itT)$ are wave operators (T is negative Laplacian for short-range potentials under an appropriate unit system, while it has to be modified when long-range potentials are included); and $\mathcal{F}$ is Fourier transformation so that $\mathcal{F}T\mathcal{F}^{-1}$ is a multiplication operator by $|\xi|^2$ in the momentum representation $L^2(\mathbf{R}^3_\xi)$. By definition, S commutes with T. This makes $\widehat{S}$ decomposable with respect to $|\xi|^2 = \mathcal{F}T\mathcal{F}^{-1}$. Namely, for *a.e.* $E > 0$, there is a unitary operator $\mathcal{S}(E)$ on $L^2(S^2)$, S^2 being two dimensional sphere with radius one, such that for *a.e.* $E > 0$ and $\omega \in S^2$

$$(\widehat{S}h)(\sqrt{E}\omega) = \left(\mathcal{S}(E)h(\sqrt{E}\cdot)\right)(\omega), \quad h \in L^2(\mathbf{R}^3_\xi) = L^2((0,\infty), L^2(S^2_\omega), |\xi|^2 d|\xi|).$$

Thus $\widehat{S}$ can be written as $\widehat{S} = \{\mathcal{S}(E)\}_{E>0}$. It is known [5] that $\mathcal{S}(E)$ can be expressed as

$$(\mathcal{S}(E)\varphi)(\theta) = \varphi(\theta) - 2\pi i\sqrt{E}\int_{S^2}\mathcal{S}(E,\theta,\omega)\varphi(\omega)d\omega$$

for $\varphi \in L^2(S^2)$. The integral kernel $\mathcal{S}(E,\theta,\omega)$ with ω being the direction of initial wave, is the scattering amplitude $\mathcal{S}(E,\theta)$ stated in the above and $|\mathcal{S}(E,\theta,\omega)|^2$ is called differential cross section. These are the most important quantities in physics in the sense that they are the *only* quantities which can be observed in actual physical observation.

The energy level E in the previous example thus corresponds to the energy shell $T = E$, and the replacement of E by E' in the above means that T is replaced by a *classical relativistic* quantity $E' = c\sqrt{p^2 + m^2c^2} - mc^2$. We have then seen that the calculation in the above gives a correct relativistic result, which explains the actual observation.

Axiom 6.1$'$ is concerned with the observation of the final stage of scattering phenomena. To include the gravity into our consideration, we extend Axiom 6.1$'$ to the intermediate process of quantum-mechanical evolution. The intermediate process cannot be an object of any *actual* observation, because the intermediate observation would change the process itself, consequently the result observed at the final stage would be altered. Our next Axiom 6.1 is an extension of Axiom 6.1$'$ from the *actual* observation to the *ideal* observation in the sense that Axiom 6.1 is concerned with such invisible intermediate processes and modifies the *ideal* intermediate classical quantities by relativistic change of coordinates. The spirit of the treatment developed below is to trace the quantum-mechanical paths by ideal observations so that the quantities will be transformed into classical quantities at each step, but the quantum-mechanical paths will *not* be altered owing to the *ideality* of the observations. The classical Hamiltonian obtained at the last step will be "requantized" to recapture the quantum-mechanical nature of the process, therefore the ideality of the intermediate observations will be realized in the final expression of the propagator of the observed system.

VI.2.2. With these remarks in mind, we return to the general k cluster case, and consider a way to include gravity in our framework.

In the scattering process into $k \geq 1$ clusters, what we observe are the centers of mass of those k clusters $C_1, \cdots, C_k$, and of the combined system $L = (L_1, \cdots, L_k)$. In the

example of the two body case of section VI.2.1, only the combined system $L = (L_1, L_2)$ appears due to $H_1 = H_2 = 0$, therefore the replacement of T by E' is concerned with the free energy between two clusters C_1 and C_2 of the combined system $L = (L_1, L_2)$.

Following this treatment of T in the section VI.2.1, we replace $T = T_b$ in the exponent of $\exp(-it_L h_b) = \exp(-it_L(T_b + I_b(x_b, 0)))$ on the right hand side of the asymptotic relation (8) by the relativistic kinetic energy T'_b among the clusters $C_1, \cdots, C_k$ around the center of mass of $L = (L_1, \cdots, L_k)$, defined by

$$T'_b = \sum_{j=1}^{k} \left(c\sqrt{p_j^2 + m_j^2 c^2} - m_j c^2 \right). \tag{10}$$

Here $m_j > 0$ is the rest mass of the cluster C_j, which involves all the internal energies like the kinetic energies inside C_j and the rest masses of the particles inside C_j, and p_j is the relativistic momentum of the center of mass of C_j inside L around the center of mass of L. For simplicity, we assume that the center of mass of L is stationary relative to the observer. Then we can set in the exponent of $\exp(-it_L(T'_b + I_b(x_b, 0)))$

$$t_L = t_O, \tag{11}$$

where t_O is the observer's time.

For the factors $\exp(-it_L H_j)$ on the right hand side of (8), the object of the *ideal* observation is the centers of mass of the k number of clusters $C_1, \cdots, C_k$. These are the ones which now require the relativistic treatment. Since we identify the clusters $C_1, \cdots, C_k$ as their centers of mass moving in a classical fashion, t_L in the exponent of $\exp(-it_L H_j)$ should be replaced by c^{-1} times the classical relativistic proper time at the origin of the local system L_j, which is equal to the quantum-mechanical local time t_j of the sub-local system L_j. By the same reason and by the fact that H_j is the internal energy of the cluster C_j relative to its center of mass, it would be justified to replace the Hamiltonian H_j in the exponent of $\exp(-it_j H_j)$ by the classical relativistic energy *inside* the cluster C_j around its center of mass

$$H'_j = m_j c^2, \tag{12}$$

where $m_j > 0$ is the same as in the above.

Summing up, we arrive at the following postulate, which has the same spirit as in Axiom 6.1' and includes Axiom 6.1' as a special case concerned with actual observation:

Axiom 6.1. In either actual or ideal observation, the space-time coordinates (ct_L, x_L) and the four momentum $p = (p^\mu) = (E_L/c, p_L)$ of the observed system L should be replaced by classical relativistic quantities, which are transformed into the classical quantities (ct_O, x_O) and $p = (E_O/c, p_O)$ in the observer's system L_O according to the relativistic change of coordinates specified in Axioms 4 and 5. Here t_L is the local time of the system L and x_L is the internal space coordinates inside the system L; and E_L is the internal energy of the system L and p_L is the momentum of the center of mass of the system L.

In the case of the present scattering process into k clusters, the system L in this axiom is each of the local systems L_j $(j = 1, 2, \cdots, k)$ and L.

14

We continue to consider the k centers of mass of the clusters $C_1, \cdots, C_k$. At the final stage of the scattering process, the velocities of the centers of mass of the clusters $C_1, \cdots, C_k$ would be steady, say $v_1, \cdots, v_k$, relative to the observer's system. Thus, according to Axiom 6.1, the local times t_j $(j = 1, 2, \cdots, k)$ in the exponent of $\exp(-it_j H'_j)$, which are equal to c^{-1} times the relativistic proper times at the origins $x_j = 0$ of the local systems L_j, are expressed in the observer's time coordinate t_O by

$$t_j = t_O \sqrt{1 - (v_j/c)^2} \approx t_O \left(1 - v_j^2/(2c^2)\right), \quad j = 1, 2, \cdots, k, \tag{13}$$

where we have assumed $|v_j/c| \ll 1$ and used Axioms 4 and 5 to deduce the Lorentz transformation:

$$t_j = \frac{t_O - (v_j/c^2)x_O}{\sqrt{1 - (v_j/c)^2}}, \quad x_j = \frac{x_O - v_j t_O}{\sqrt{1 - (v_j/c)^2}}.$$

(For simplicity, we wrote the Lorentz transformation for the case of 2-dimensional space-time.)

Inserting (10), (11), (12) and (13) into the right-hand side of (8), we obtain a classical approximation of the evolution:

$$\exp\left(-it_O[(T'_b + I_b(x_b, 0) + H'_1 + \cdots + H'_k) - (m_1 v_1^2/2 + \cdots + m_k v_k^2/2)]\right) \tag{14}$$

under the assumption that $|v_j/c| \ll 1$ for all $j = 1, 2, \cdots, k$.

What we want to clarify is the final stage of the scattering process. Thus as we have mentioned, we may assume that all clusters $C_1, \cdots, C_k$ are far away from any of the other clusters and moving almost in steady velocities $v_1, \cdots, v_k$ relative to the observer. We denote by r_{ij} the distance between two centers of mass of the clusters C_i and C_j for $1 \le i < j \le k$. Then, according to our spirit that we are observing the behavior of the centers of mass of the clusters $C_1, \cdots, C_k$ in *classical* fashion following Axioms 4 and 5, the clusters $C_1, \cdots, C_k$ can be regarded to have gravitation among them. This gravitation can be calculated if we assume Einstein's field equation, $|v_j/c| \ll 1$, and certain conditions that the gravitation is weak (see [11], section 17.4), in addition to our Axioms 4 and 5. As an approximation of the first order, we obtain the gravitational potential of Newtonian type for, e.g., the pair of the clusters C_1 and $U_1 = \bigcup_{i=2}^{k} C_i$:

$$-G \sum_{i=2}^{k} m_1 m_i / r_{1i},$$

where G is Newton's gravitational constant.

Considering the k body classical problem for the k clusters $C_1, \cdots, C_k$ moving in the sum of these gravitational fields, we see that the sum of the kinetic energies of $C_1, \cdots, C_k$ and the gravitational potentials among them is constant by the classical law of conservation of energy:

$$m_1 v_1^2/2 + \cdots + m_k v_k^2/2 - G \sum_{1 \le i < j \le k} m_i m_j / r_{ij} = \text{constant}.$$

15

Assuming that $v_j \to v_{j\infty}$ as time tends to infinity, we have constant $= m_1 v_{1\infty}^2/2 + \cdots + m_k v_{k\infty}^2/2$. Inserting this relation into (14) in the above, we obtain the following as a classical approximation of the evolution (8):

$$\exp\left(-it_O\left[T_b' + I_b(x_b, 0) + \sum_{j=1}^{k}(m_j c^2 - m_j v_{j\infty}^2/2) - G\sum_{1 \leq i < j \leq k} m_i m_j/r_{ij}\right]\right). \tag{15}$$

What we do at this stage are *ideal* observations, and these observations should not give any sharp classical values. Thus we have to consider (15) as a *quantum-mechanical evolution* and we have to recapture the quantum-mechanical feature of the process. To do so we replace p_j in T_b' in (15) by a quantum-mechanical momentum D_j, where D_j is a differential operator $-i\frac{\partial}{\partial x_j} = -i\left(\frac{\partial}{\partial x_{j1}}, \frac{\partial}{\partial x_{j2}}, \frac{\partial}{\partial x_{j3}}\right)$ with respect to the 3-dimensional coordinates x_j of the center of mass of the cluster C_j. Thus the actual process should be described by (15) with T_b' replaced by a quantum-mechanical Hamiltonian

$$\widetilde{T}_b = \sum_{j=1}^{k}\left(c\sqrt{D_j^2 + m_j^2 c^2} - m_j c^2\right).$$

This procedure may be called "requantization," and is summarized as the following axiom concerning the ideal observation.

Axiom 6.2. In the expression describing the classical process at the time of the *ideal* observation, the intercluster momentum $p_j = (p_{j1}, p_{j2}, p_{j3})$ should be replaced by a quantum-mechanical momentum $D_j = -i\left(\frac{\partial}{\partial x_{j1}}, \frac{\partial}{\partial x_{j2}}, \frac{\partial}{\partial x_{j3}}\right)$. Then this gives the evolution describing the intermediate *quantum-mechanical* process.

We thus arrive at an approximation for a quantum-mechanical Hamiltonian including gravitational effect up to a constant term, which depends on the system L and its decomposition into $L_1, \cdots, L_k$, but not affecting the quantum-mechanical evolution, therefore can be eliminated:

$$\widetilde{H}_L = \widetilde{T}_b + I_b(x_b, 0) - G\sum_{1 \leq i < j \leq k} m_i m_j/r_{ij}$$

$$= \sum_{j=1}^{k}\left(c\sqrt{D_j^2 + m_j^2 c^2} - m_j c^2\right) + I_b(x_b, 0) - G\sum_{1 \leq i < j \leq k} m_i m_j/r_{ij}. \tag{16}$$

We remark that the gravitational terms here come from the substitution of local times t_j to the time t_L in the factors $\exp(-it_L H_j)$ on the right-hand side of (8). This form of Hamiltonian in (16) is actually used in [10] with $I_b = 0$ to explain the stability and instability of cold stars of large mass, showing the effectiveness of the Hamiltonian.

Summarizing these arguments from (8) to (16), we have obtained the following *interpretation* of the observation of the quantum-mechanical evolution: To get our prediction

for the observation of local systems $L_1, \cdots, L_k$, the quantum-mechanical evolution of the combined local system $L = (L_1, \cdots, L_k)$

$$\exp(-it_L H_L) f$$

should be replaced by the following evolution, in the approximation of the first order under the assumption that $|v_j/c| \ll 1$ $(j = 1, 2, \cdots, k)$ and the gravitation is weak,

$$(\exp(-it_O \widetilde{H}_L) \otimes \underbrace{I \otimes \cdots \otimes I}_{k \text{ factors}}) P_b \Omega_b^{+*} f, \tag{17}$$

provided that the original evolution $\exp(-it_L H_L) f$ decomposes into k number of clusters $C_1, \cdots, C_k$ as $t_L \to \infty$ in the sense of (8). Here b is the cluster decomposition $b = (C_1, \cdots, C_k)$ that corresponds to the decomposition $L = (L_1, \cdots, L_k)$ of L; t_O is the observer's time; and

$$\widetilde{H}_L = \widetilde{T}_b + I_b(x_b, 0) - G \sum_{1 \leq i < j \leq k} m_i m_j / r_{ij} \tag{18}$$

is the relativistic Hamiltonian inside L given by (16), which describes the motion of the centers of mass of the clusters $C_1, \cdots, C_k$.

We remark that (17) may produce a bound state combining $C_1, \cdots, C_k$ as $t_O \to \infty$ therefore for all t_O, due to the gravitational potentials in the exponent. Note that this is not prohibited by our assumption that $\exp(-it_L H_L) f$ has to decompose into k clusters $C_1, \cdots, C_k$, because the assumption is concerned with the original Hamiltonian H_L but not with the resultant Hamiltonian $\widetilde{H}_L$.

Extending our primitive assumption Axiom 6.1′, which was valid for an example stated in section VI.2.1, we have arrived at a relativistic Hamiltonian $\widetilde{H}_L$, which would describe approximately the intermediate process, under the assumption that the gravitation is weak and the velocities of the particles are small compared to c, by using the Lorentz transformation. We note that, since we started our argument from the asymptotic relation (8), which is concerned with the final stage of scattering processes, we could assume that the velocities of particles are almost steady relative to the observer in the correspondent classical expressions of the processes, therefore we could appeal to the Lorentz transformations when performing the change of coordinates in the relevant arguments.

The final values of scattering amplitude should be calculated by using the Hamiltonian $\widetilde{H}_L$. Then they would explain actual observations. This is our prediction for the observation of relativistic quantum-mechanical phenomena including the effects by gravity and quantum-mechanical forces.

In the example discussed in section VI.2.1, this approach gives the same result as (9) in the approximation of the first order, showing the consistency of our spirit (see [9]).

VII. Existence of Local Motion

We are in a position to see how the stationary nature of the universe and the existence of local motion and hence local time are compatibly incorporated into our formulation.

VII.1. Gödel's theorem

Our starting point is the incompleteness theorem proved by Gödel [4]. It states that any consistent formal theory that can describe number theory includes an infinite number of undecidable propositions. The physical world includes at least natural numbers, and it is described by a system of words, which can be translated into a formal physics theory. The theory of physics, if consistent, therefore includes an undecidable proposition, i.e. a proposition whose correctness cannot be known by human beings until one finds a phenomenon or observation that supports the proposition or denies the proposition. Such propositions exist infinitely according to Gödel's theorem. Thus human beings, or any other finite entity, will never be able to reach a "final" theory that can express the totality of the phenomena in the universe.

Thus we have to assume that any human observer sees a part or subsystem L of the universe and never gets the total Hamiltonian H in (1) by his observation. Here the total Hamiltonian H is an *ideal* Hamiltonian that might be gotten by "God." In other words, a consequence from Gödel's theorem is that the Hamiltonian that an observer assumes with his observable universe is a part H_L of H. Stating explicitly, the consequence from Gödel's theorem is the following proposition

$$H = H_L + I + H_E, \quad H_E \neq 0, \tag{19}$$

where H_E is an unknown Hamiltonian describing the system E exterior to the realm of the observer, whose existence, i.e. $H_E \neq 0$, is assured by Gödel's theorem. This unknown system E includes all that is unknown to the observer. E.g., it might contain particles which exist near us but have not been discovered yet, or are unobservable for some reason at the time of observation. The term I is an unknown interaction between the observed system L and the unknown system E. Since the exterior system E is assured to exist by Gödel's theorem, the interaction I does not vanish: In fact assume I vanishes. Then the observed system L and the exterior system E do not interact, which is the same as that the exterior system E does not exist for the observer. On the other hand, assigning the so-called Gödel number to each proposition in number theory, Gödel constructs undecidable propositions in number theory by a diagonal argument, which shows that any consistent formal theory has a region exterior to the knowable world (see [4]). Thus the observer must be able to construct a proposition by Gödel's procedure that proves E exists, which means $I \neq 0$. By the same reason, I is not a constant operator:

$$I \neq \text{constant operator.} \tag{20}$$

For suppose it is a constant operator. Then the systems L and E do not change no matter how far or how near they are located because the interaction between L and E

is a constant operator. This is the same situation as that the interaction does not exist, thus reduces to the case $I = 0$ above.

We now arrive at the following observation: For an observer, the observable universe is a part L of the total universe and it looks as though it follows the Hamiltonian H_L, not following the total Hamiltonian H. And the state of the system L is described by a part $\phi(\cdot, y)$ of the state ϕ of the total universe, where y is an unknown coordinate of system L inside the total universe, and $\cdot$ is the variable controllable by the observer, which we will denote by x.

VII.2. Local Time Exists

In the following argument, we assume an exact relation:

$$H\phi = 0 \tag{21}$$

instead of (1), for simplicity.

Assume now, as is usually expected under condition (21), that there is no local time of L, i.e. that the state $\phi(x, y)$ is an eigenstate of the local Hamiltonian H_L for some $y = y_0$ and a real number μ:

$$H_L\phi(x, y_0) = \mu\phi(x, y_0). \tag{22}$$

Then from (19), (21) and (22) follows that

$$\begin{aligned}
0 = H\phi(x, y_0) &= H_L\phi(x, y_0) + I(x, y_0)\phi(x, y_0) + H_E\phi(x, y_0) \\
&= (\mu + I(x, y_0))\phi(x, y_0) + H_E\phi(x, y_0).
\end{aligned} \tag{23}$$

Here x varies over the possible positions of the particles inside L. On the other hand, since H_E is the Hamiltonian describing the system E exterior to L, it does not affect the variable x and acts only on the variable y. Thus $H_E\phi(x, y_0)$ varies as a bare function $\phi(x, y_0)$ insofar as the variable x is concerned. Equation (23) is now written: For all x

$$H_E\phi(x, y_0) = -(\mu + I(x, y_0))\phi(x, y_0). \tag{24}$$

As we have seen in (20), the interaction I is not a constant operator and varies when x varies[1], whereas the action of H_E on ϕ does not. Thus there is a nonempty set of points x_0 where $H_E\phi(x_0, y_0)$ and $-(\mu + I(x_0, y_0))\phi(x_0, y_0)$ are different, and (24) does not hold at such points x_0. If I is assumed to be continuous in the variables x and y, these points x_0 constitutes a set of positive measure. This then implies that our assumption (22) is wrong. Thus a subsystem L of the universe cannot be a bound state with respect to the observer's Hamiltonian H_L. This means that the system L is observed as a non-stationary system, therefore there must be observed a motion inside the system L. This proves that the "time" of the local system L *exists for the observer* as a measure of motion, whereas the total universe is stationary and does not have "time."

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[1]Note that Gödel's theorem applies to any fixed $y = y_0$ in (20). Namely, for any position y_0 of the system L in the universe, the observer must be able to know that the exterior system E exists because Gödel's theorem is a universal statement valid throughout the universe. Hence $I(x, y_0)$ is not a constant operator with respect to x for any fixed y_0.

VII.3. A refined argument

To show the argument in section VII.2 more explicitly, we consider a simple case of

$$H = \frac{1}{2} \sum_{k=1}^{N} h^{ab}(X_k) p_{ka} p_{kb} + V(X).$$

Here N $(1 \leq N \leq \infty)$ is the number of particles in the universe, h^{ab} is a three-metric, $X_k \in R^3$ is the position of the k-th particle, p_{ka} is a functional derivative corresponding to momenta of the k-th particle, and $V(X)$ is a potential. The configuration $X = (X_1, X_2, \cdots, X_N)$ of total particles is decomposed as $X = (x, y)$ accordingly to if the k-th particle is inside L or not, i.e. if the k-th particle is in L, X_k is a component of x and if not it is that of y. H is decomposed as follows:

$$H = H_L + I + H_E.$$

Here H_L is the Hamiltonian of a subsystem L that acts only on x, H_E is the Hamiltonian describing the exterior E of L that acts only on y, and $I = I(x, y)$ is the interaction between the systems L and E. Note that H_L and H_E commute.

Theorem 3 *Let P denote the eigenprojection onto the space of all bound states of H. Let P_L be the eigenprojection for H_L. Then we have*

$$(1 - P_L)P \neq 0, \tag{25}$$

unless the interaction $I = I(x, y)$ is a constant with respect to x for any y.

Proof. Assume that (25) is incorrect. Then we have

$$P_L P = P.$$

Taking the adjoint operators on the both sides, we then have

$$P P_L = P.$$

Thus $[P_L, P] = P_L P - P P_L = 0$. But in generic this does not hold because

$$[H_L, H] = [H_L, H_L + I + H_E] = [H_L, I] \neq 0,$$

unless $I(x, y)$ is equal to a constant with respect to x. Q.E.D.

Remark. In the context of section V, the theorem implies the following:

$$(1 - P_L)P\mathcal{U} \neq \{0\},$$

where $\mathcal{U}$ is a Hilbert space consisting of all possible states ϕ of the total universe. This relation implies that there is a vector $\phi \neq 0$ in $\mathcal{U}$ which satisfies $H\phi = \lambda\phi$ for a real number λ while $H_L \Phi \neq \mu\Phi$ for any real number μ, where $\Phi = \phi(\cdot, y)$ is a state vector of the subsystem L with an appropriate choice of the position y of the subsystem. Thus the space generated by $\phi(\cdot, y)$'s when y varies is non-trivial in the sense of Definition 1 in section V, which proves for the universe ϕ that any local system L is non-trivial, and hence proves the existence of local time for any local system of the universe ϕ. Thus we have at least one stationary universe ϕ where every local system has its local time.

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